# Spectral Domain Analysis of Frequency Dependent Propogation Characteristics of Planar Structures on Uniaxial Medium

HYUCKJAE LEE, STUDENT MEMBER, IEEE, AND VIJAI K. TRIPATHI, MEMBER, IEEE

*Abstract* —**The propagation characteristics of single and multilayered uniaxial dielectric waveguides and planar structures on uniaxial medium can be determined by utilizing Hertzian potentials along the optical axis. The electric and magnetic Hertzian potentials, having components along the optical axis only, lead to TM and TE modes, respectively, with respect to that axis. The dyadic Green's function in Fourier transform domain (immittance matrix) required to solve for the propagation characteristics of planar structures on uniaxial medium are derived for all three orientations of the optical axis. The immittance matrix for all three cases is in the same form as that for the isotropic medium and hence the known Galerkin's method can be used to solve for the propagation characteristics of the structure.**

## I. Introduction

THE STUDY OF the propagation characteristics of planar inhomogeneous structures on anisotropic substrates has been confined mostly to the case of quasi-TEM analysis of single or coupled microstrip lines [1]–[10] since for this case, the affine transformation renders the problem to the familiar one of isotropic medium. In most cases of the inhomogeneous guided wave systems with anisotropic medium, the formulation of the frequency dependent boundary value problem becomes quite complicated. Among many anisotropic materials the uniaxial dielectric, and gyromagnetic substrates are of interest for various applications at microwave and higher frequencies. In this paper, the full wave analysis for an inhomogeneous uniaxial medium is explicitly formulated. The results can be used either to study the effect of such anisotropy on the properties of planar structures on single or multilayered uniaxial dielectric medium since some of the substrates used such as sapphire, Epsilon-10, Polytetrafluoroethylene, and others [1], [11] are uniaxial, or to study the structures for various applications including electrooptic modulators [4], [10], [12], and equalization of even- and odd-mode phase velocities for symmetrical coupled microstrip lines [7], [8].

Examples of such structures include single and coupled microstrip lines, slot lines, coplanar lines and waveguides, and fin lines. Some of the structures such as microstrip lines do support a quasi-TEM mode at lower frequencies and become dispersive at higher frequencies, whereas others such as slot lines can only support hybrid or higher order modes.

One of the most efficient and accurate computational methods for determining the properties of such structures on isotropic medium is the use of the immittance function, and Galerkin's method in Fourier transform domain [13]–[18]. It is shown in this paper that the immittance matrix function for such structures on a single uniaxial substrate or multilayered uniaxial and isotropic or other uniaxial medium can be derived in exactly the same form as the one for isotropic medium rendering itself to computational solutions by utilizing the same Galerkin's method. It is found that the problem can be most conveniently formulated in terms of both magnetic and electric Hertzian potentials having components along the optical axis only. This corresponds to constructing the solutions for field components in terms of TE and TM modes with respect to the optical axis. Otherwise, the problem becomes similar to that of general biaxial medium which is hardly tractable [19]. The immittance matrix for all three cases of the orientation of the optical axis are derived in Section III for the case of inhomogeneous single layer medium after a brief review of the formulation of the boundary value problem at hand in terms of the Hertzian potentials in Section II. The immittance matrix derived for all three cases of the orientation of the optical axis reduces to the known expressions if the medium is assumed to be isotropic and leads to the static Green's function from affine transformation for the static boundary value problem.

## II. Formulation of the Problem in Terms of the Hertzian Potential Functions

The solution for all the electric and magnetic field components required to satisfy the boundary conditions at the interface of uniaxial media with other isotropic or uniaxial media (with or without filament sources at the interface) can be derived from the electric and magnetic Hertzian potentials having components along the optical axis only. The magnetic Hertzian potential along the optic axis gives a solution for electric field $\boldsymbol{E}$ with no component along that direction while the electric Hertzian potential directed along the optic axis leads to a solution for magnetic field $\boldsymbol{H}$ having no uniaxial direction component. Starting from the Maxwell's equations in source-free regions characterized by $\mu=\mu_0$ and

$$\hat{\epsilon}=\begin{bmatrix}\epsilon_x & 0 & 0\\ 0 & \epsilon_y & 0\\ 0 & 0 & \epsilon_z\end{bmatrix}$$

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The authors are with the Department of Electrical and Computer Engineering, Oregon State University, Corvallis, OR 97331.

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with two of the diagonal terms equal for the uniaxial medium depending on the orientation, the fields are derived in terms of the Hertzian potentials directed along the optical axis [20]. Let $\hat{a}_\xi$ be the unit vector along the optical axis $\xi$($\xi$ is $x$ or $y$, or $z$) with a dielectric constant $\epsilon_1$ along that axis and $\epsilon_2$ along the two axes normal to the optical axis. Then, for harmonically oscillating field case ($e^{j\omega t}$ variation), the electric and magnetic fields can be found from [20].

a) The ordinary wave is found from magnetic Hertzian potential $\boldsymbol{\pi}_h$ as

$$\boldsymbol{E} = -j\omega\mu_0\nabla\times\boldsymbol{\pi}_h \tag{1}$$

$$\boldsymbol{H} = \nabla\times\nabla\times\boldsymbol{\pi}_h \tag{2}$$

where $\boldsymbol{\pi}_h = \pi_h \hat{a}_\xi$ is the solution of

$$\nabla^2\pi_h + \omega^2\mu_0\epsilon_2\pi_h = 0. \tag{3}$$

b) The extraordinary wave is found from electric Hertzian potential $\boldsymbol{\pi}_e$ as

$$\boldsymbol{E} = \omega^2\mu_0\epsilon_0\boldsymbol{\pi}_e + \frac{\epsilon_0}{\epsilon_2}\nabla\nabla\cdot\boldsymbol{\pi}_e \tag{4}$$

$$\boldsymbol{H} = j\omega\epsilon_0\nabla\times\boldsymbol{\pi}_e \tag{5}$$

where $\boldsymbol{\pi}_e = \pi_e\hat{a}_\xi$ is the solution of

$$\nabla^2\pi_e + \omega^2\mu_0\epsilon_1\pi_e + \frac{\epsilon_1-\epsilon_2}{\epsilon_2}\frac{\partial^2\pi_e}{\partial\xi^2} = 0. \tag{6}$$

From the above equations we see that the magnetic Hertzian potential along the optical axis which is the solution of (3) gives solutions with electric field $\boldsymbol{E}$ having no component along the optical axis while the electric Hertzian potential along the optical axis gives a solution for magnetic field $\boldsymbol{H}$ with no components along that direction.

For a given uniform guided wave system along the $z$-direction, we can assume an $e^{-j\beta z}$ type variation along that direction. The dispersion characteristics for that system then are to be found by applying the boundary conditions for electric and magnetic fields derived from (1) through (6). For cylindrical dielectric waveguides with the optical axis along the $z$-direction [21] and slab guides with the optical axis along any of the three directions, $\boldsymbol{E}$ and $\boldsymbol{H}$ are readily found from (1), (2), (4), and (5) in terms of $\boldsymbol{\pi}_e$ and $\boldsymbol{\pi}_h$, which are the solutions of (3) and (6). Applying the boundary conditions for tangential and normal components of the fields then leads to the dispersion equation for various modes. For planar structures on an inhomogeneous uniaxial medium, the solution leads to the expression for the tangential components of the electric fields at the boundaries in terms of the unknown current sources. In the Fourier transform domain this relationship is expressed in terms of the desired impedance functions which can then be used to compute the propagation characteristics by using Galerkin's method.

It should be noted that for general multilayered structures with different orientations of the optical axis in each region the procedure calls for constructing the solution for fields in terms of the two Hertzian potentials defined along the optical axis for each region.

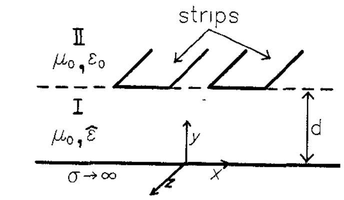


Fig. 1. Planar structure on an uniaxial medium.

## III. Spectral Domain Analysis

In the Fourier transform domain, all the variables are transformed with respect to $x$ according to

$$\tilde{\psi}(\alpha, y) = \int_{-\infty}^{\infty}\psi(x, y)e^{-j\alpha x}dx \tag{7}$$

then

$$\psi(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\psi}(\alpha, y)e^{jx\alpha}d\alpha.$$

The immittance matrix function for the structures is then derived by writing the solutions for the field components and applying the boundary conditions. For example, the desired impedance matrix sought with filament current sources at one interface ($y = d$) only is given as

$$\begin{bmatrix}\tilde{E}_x(\alpha, d)\\ \tilde{E}_z(\alpha, d)\end{bmatrix} = \begin{bmatrix}\tilde{Z}_{xx}(\alpha,\beta,d) & \tilde{Z}_{xz}(\alpha,\beta,d)\\ \tilde{Z}_{zx}(\alpha,\beta,d) & \tilde{Z}_{zz}(\alpha,\beta,d)\end{bmatrix}\begin{bmatrix}\tilde{J}_x(\alpha, d)\\ \tilde{J}_z(\alpha, d)\end{bmatrix}. \tag{8}$$

This matrix function is derived in this section for lines deposited on a single uniaxial medium of height $d$ with a ground plane, e.g., single or multiple coupled microstrip lines, for the three cases of the orientation of the optical axis in Fig. 1.

### A. Optical Axis Normal to the Interface ($\epsilon_y = \epsilon_1, \epsilon_x = \epsilon_z = \epsilon_2$)

For this case, $\boldsymbol{\pi}_h = \pi_h\hat{a}_y$ and $\boldsymbol{\pi}_e = \pi_e\hat{a}_y$, and in the Fourier transform domain, $\tilde{\pi}_e$, $\tilde{\pi}_h$ and the tangential field components are the solutions of the following equations in the dielectric and air medium.

*1) Uniaxial Medium (Region I), $y < d$:*

$$\frac{d^2\tilde{\pi}_{h1}(\alpha, y)}{dy^2} - \gamma_h^2\tilde{\pi}_{h1}(\alpha, y) = 0 \tag{9a}$$

$$\frac{d^2\tilde{\pi}_{e1}(\alpha, y)}{dy^2} - \gamma_e^2\tilde{\pi}_{e1}(\alpha, y) = 0 \tag{9b}$$

$$\tilde{E}_{x1}(\alpha, y) = -j\alpha\frac{\epsilon_0}{\epsilon_2}\frac{d\tilde{\pi}_{e1}}{dy} + \omega\mu_0\beta\tilde{\pi}_{h1} \tag{10a}$$

$$\tilde{E}_{z1}(\alpha, y) = -j\beta\frac{\epsilon_0}{\epsilon_2}\frac{d\tilde{\pi}_{e1}}{dy} - \omega\mu_0\alpha\tilde{\pi}_{h1} \tag{10b}$$

$$\tilde{H}_{x1}(\alpha, y) = -\omega\epsilon_0\beta\tilde{\pi}_{e1} - j\alpha\frac{d\tilde{\pi}_{h1}}{dy} \tag{10c}$$

$$\tilde{H}_{z1}(\alpha, y) = \omega\alpha\epsilon_0\tilde{\pi}_{e1} - j\beta\frac{d\tilde{\pi}_{h1}}{dy}. \tag{10d}$$

*2) Air Medium (Region II), $y > d$:*

$$\frac{d^2\tilde{\pi}_{h2}(\alpha, y)}{dy^2} - \gamma_0^2\tilde{\pi}_{h2}(\alpha, y) = 0 \tag{11a}$$

$$\frac{d^2\tilde{\pi}_{e2}(\alpha, y)}{dy^2} - \gamma_0^2\tilde{\pi}_{e2}(\alpha, y) = 0 \tag{11b}$$

$$\tilde{E}_{x2}(\alpha, y) = -j\alpha\frac{d\tilde{\pi}_{e2}}{dy} + \omega\mu_0\beta\tilde{\pi}_{h2} \tag{12a}$$

$$\tilde{E}_{z2}(\alpha, y) = -j\beta\frac{d\tilde{\pi}_{e2}}{dy} - \omega\mu_0\alpha\tilde{\pi}_{h2} \tag{12b}$$

$$\tilde{H}_{x2}(\alpha, y) = -\omega\epsilon_0\beta\tilde{\pi}_{e2} - j\alpha\frac{d\tilde{\pi}_{h2}}{dy} \tag{12c}$$

$$\tilde{H}_{z2}(\alpha, y) = \omega\epsilon_0\alpha\tilde{\pi}_{e2} - j\beta\frac{d\tilde{\pi}_{h2}}{dy} \tag{12d}$$

where

$$k_0^2 \triangleq \omega^2\mu_0\epsilon_0$$

$$k_1^2 \triangleq \omega^2\mu_0\epsilon_1$$

$$k_2^2 \triangleq \omega^2\mu_0\epsilon_2$$

$$\gamma_0^2 \triangleq \alpha^2 + \beta^2 - k_0^2$$

$$\gamma_h^2 \triangleq \alpha^2 + \beta^2 - k_2^2$$

and

$$\gamma_e^2 \triangleq \frac{\epsilon_2}{\epsilon_1}(\alpha^2 + \beta^2 - k_1^2).$$

The solutions for potentials as given by (9) and (11) are

$$\tilde{\pi}_{h1}(\alpha, y) = A(\alpha)\sinh\gamma_h y + A'(\alpha)\cosh\gamma_h y \tag{13a}$$

$$\tilde{\pi}_{e1}(\alpha, y) = C(\alpha)\cosh\gamma_e y + C'(\alpha)\sinh\gamma_e y \tag{13b}$$

$$\tilde{\pi}_{h2}(\alpha, y) = B(\alpha)e^{-\gamma_0(y-d)} \tag{13c}$$

$$\tilde{\pi}_{e2}(\alpha, y) = D(\alpha)e^{-\gamma_0(y-d)}. \tag{13d}$$

Substituting these into expressions for field components as given by (10) and (12) and applying the boundary conditions as given by

$$\hat{a}_y \times \boldsymbol{E}_1 = 0, \quad \text{at } y = 0 \tag{14a}$$

$$\hat{a}_y \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0, \quad \text{at } y = d \tag{14b}$$

$$\hat{a}_y \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}, \quad \text{the surface current density at } y = d \tag{14c}$$

leads to $A'(\alpha) = C'(\alpha) = 0$ from (14a) and the four equations for $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, and $D(\alpha)$ from (14b) and (14c) in terms of $\tilde{J}_x(\alpha, d)$ and $\tilde{J}_z(\alpha, d)$ as given by

$$\begin{bmatrix} -\alpha\omega\mu_0\sinh\gamma_h d & \alpha\omega\mu_0 & -j\beta\frac{\epsilon_0}{\epsilon_2}\gamma_e\sinh\gamma_e d & -j\beta\gamma_0 \\ \omega\mu_0\beta\sinh\gamma_h d & -\omega\mu_0\beta & -j\alpha\frac{\epsilon_0}{\epsilon_2}\gamma_e\sinh\gamma_e d & -j\alpha\gamma_0 \\ -j\beta\gamma_h\cosh\gamma_h d & -j\beta\gamma_0 & \alpha\omega\epsilon_0\cosh\gamma_e d & -\alpha\omega\epsilon_0 \\ -j\alpha\gamma_h\cosh\gamma_h d & -j\alpha\gamma_0 & -\omega\epsilon_0\beta\cosh\gamma_e d & \omega\epsilon_0\beta \end{bmatrix} \begin{bmatrix} A(\alpha) \\ B(\alpha) \\ C(\alpha) \\ D(\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\tilde{J}_x(\alpha, d) \\ \tilde{J}_z(\alpha, d) \end{bmatrix}. \tag{15}$$

The above equations are much easier to manipulate than the ones derived for the other two orientations of the optical axis to be considered after this section, and are readily solved for the coefficients $A(\alpha)$–$D(\alpha)$ to give

$$A(\alpha) = \frac{1}{\sinh\gamma_h d}B(\alpha) \tag{16}$$

$$C(\alpha) = -\frac{\epsilon_2}{\epsilon_0}\frac{\gamma_0}{\gamma_e\sinh\gamma_e d}D(\alpha) \tag{17}$$

$$B(\alpha) = \frac{-j\beta\tilde{J}_x(\alpha, d) + j\alpha\tilde{J}_z(\alpha, d)}{(\alpha^2+\beta^2)(\gamma_0 + \gamma_h\coth\gamma_h d)} \tag{18}$$

$$D(\alpha) = \frac{\alpha\tilde{J}_x(\alpha, d) + \beta\tilde{J}_z(\alpha, d)}{\omega(\alpha^2+\beta^2)\left[\epsilon_0 + \epsilon_2\frac{\gamma_0}{\gamma_e}\coth\gamma_e d\right]}. \tag{19}$$

It should be noted from (16) and (17) that for this case the two TE and TM modes with respect to the $y$-axis are decoupled as is the case for isotropic medium inherently implied in Itoh's analysis of such structures utilizing the transverse equivalent transmission line method for multiple layer isotropic medium [18], and that decoupling is the merit unique to this orientation.

The electric field components at the interface, $y = d$, are given by

$$\tilde{E}_x(\alpha, d) = \omega\mu_0\beta B(\alpha) + j\alpha\gamma_0 D(\alpha) \tag{20a}$$

$$\tilde{E}_z(\alpha, d) = -\alpha\omega\mu_0 B(\alpha) + j\beta\gamma_0 D(\alpha). \tag{20b}$$

Substituting for $B(\alpha)$ and $D(\alpha)$ from (18) and (19) leads to the elements of the impedance matrix defined by (8) and are given by

$$\tilde{Z}_{xx}(\alpha, \beta, d) = \frac{j}{\omega(\alpha^2+\beta^2)}\left[\frac{-\omega^2\mu_0\beta^2}{g_1} + \frac{\alpha^2\gamma_0\gamma_e}{g_2}\right] \tag{21a}$$

$$\tilde{Z}_{xz}(\alpha, \beta, d) = \tilde{Z}_{zx}(\alpha, \beta, d) = \frac{j}{\omega(\alpha^2+\beta^2)}\left[\frac{\omega^2\mu_0\alpha\beta}{g_1} + \frac{\alpha\beta\gamma_0\gamma_e}{g_2}\right] \tag{21b}$$

$$\tilde{Z}_{zz}(\alpha, \beta, d) = \frac{j}{\omega(\alpha^2+\beta^2)}\left[\frac{-\omega^2\mu_0\alpha^2}{g_1} + \frac{\beta^2\gamma_0\gamma_e}{g_2}\right] \tag{21c}$$

where

$$g_1 \triangleq \gamma_0 + \gamma_h\coth\gamma_h d$$

$$g_2 \triangleq \epsilon_0\gamma_e + \epsilon_2\gamma_0\coth\gamma_e d.$$

The above equations for the impedance functions reduce to those for the isotropic medium case when $\epsilon_1 = \epsilon_2 = \epsilon_0\epsilon_r$, and lead to the Green's function from the known affine transformation for the static case.

For the isotropic medium we see that

$$\gamma_e^2 = \gamma_h^2 = \alpha^2 + \beta^2 - k_1^2$$

and

$$\gamma_0^2 = \alpha^2 + \beta^2 - k_0^2.$$

Then, the matrix elements $\tilde{Z}_{xx}$, $\tilde{Z}_{xz}$, and $\tilde{Z}_{zz}$ reduce to the ones obtained by Itoh [18].

For the static case, as $\omega \to 0$, we get

$$\gamma_h^2 = \gamma_0^2 = \alpha^2 \text{ and } \gamma_e^2 = \frac{\epsilon_2}{\epsilon_1}\alpha^2.$$

Considering a surface charge distribution as given by $\rho = -(\nabla \cdot \boldsymbol{J})/j\omega$ and potential $\tilde{\phi}(\alpha, d) = \tilde{E}_x(\alpha, d)/j\alpha$ [22] leads to the static Green's function in Fourier transform domain. The potential $\tilde{\phi}(\alpha, d)$ is found to be

$$(\alpha, d) = \left[ \frac{j/\alpha}{1 + \sqrt{\epsilon_1 \epsilon_2} \coth \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, \alpha d} \right] \tilde{\rho}(\alpha, d). \quad (22)$$

The expression in square brackets is the Green's function for planar structures such as open microstrip lines on a uniaxial medium having the optical axis normal to the interface and translates the problem to the isotropic case with the known affine transformation $\epsilon = \sqrt{\epsilon_1 \epsilon_2}$ and $y' = \sqrt{\epsilon_2/\epsilon_1}$ [4].

### B. Optical Axis Along x-Direction ($\epsilon_x = \epsilon_1$, $\epsilon_y = \epsilon_z = \epsilon_2$)

For this case $\boldsymbol{\pi}_h = \pi_h \hat{a}_x$ and $\boldsymbol{\pi}_e = \pi_e \hat{a}_x$. The tangential field components are found from (1) through (6) and are given by the following equations in the Fourier transform domain.

*1) In Region I, $y < d$:* From $\tilde{\pi}_{h1}(\alpha, y) = A(\alpha) \cosh\gamma_h y$ and $\tilde{\pi}_{e1}(\alpha, y) = C(\alpha) \sinh\gamma_e y$

$$\tilde{E}_{x1}(\alpha, y) = C(\alpha)\left(k_0^2 - \frac{\epsilon_0}{\epsilon_2}\alpha^2\right) \sinh \gamma_e y \quad (23a)$$

$$\tilde{E}_{z1}(\alpha, y) = A(\alpha)\omega\mu_0\gamma_h \sinh \gamma_h y - C(\alpha)\alpha\beta\frac{\epsilon_0}{\epsilon_2} \sinh \gamma_e y \quad (23b)$$

$$\tilde{H}_{x1}(\alpha, y) = A(\alpha)(k_2^2 - \alpha^2) \cosh \gamma_h y \quad (23c)$$

$$\tilde{H}_{z1}(\alpha, y) = -A(\alpha)\alpha\beta \cosh \gamma_h y - C(\alpha) j\omega\epsilon_0\gamma_e \cosh \gamma_e y. \quad (23d)$$

*2) In Region II, $y > d$:* From $\tilde{\pi}_{h2}(\alpha, y) = B(\alpha)e^{-\gamma_0(y-d)}$ and $\tilde{\pi}_{e2}(\alpha, y) = D(\alpha)e^{-\gamma_0(y-d)}$

$$\tilde{E}_{x2}(\alpha, y) = D(\alpha)(k_0^2 - \alpha^2)e^{-\gamma_0(y-d)} \quad (24a)$$

$$\tilde{E}_{z2}(\alpha, y) = -B(\alpha) j\omega\mu_0\gamma_0 e^{-\gamma_0(y-d)} - D(\alpha)\alpha\beta e^{-\gamma_0(y-d)} \quad (24b)$$

$$\tilde{H}_{x2}(\alpha, y) = B(\alpha)(k_0^2 - \alpha^2)e^{-\gamma_0(y-d)} \quad (24c)$$

$$\tilde{H}_{z2}(\alpha, y) = -B(\alpha)\alpha\beta e^{-\gamma_0(y-d)} + D(\alpha) j\omega\epsilon_0\gamma_0 e^{-\gamma_0(y-d)} \quad (24d)$$

where for this case

$$\gamma_h^2 \triangleq \alpha^2 + \beta^2 - k_2^2,\ \gamma_e^2 \triangleq \frac{\epsilon_1}{\epsilon_2}\alpha^2 + \beta^2 - k_1^2$$

and

$$\gamma_0^2 \triangleq \alpha^2 + \beta^2 - k_0^2.$$

Applying the boundary conditions as given by (14b) and (14c) leads to the following equation for the coefficients $A(\alpha)$–$D(\alpha)$:

$$\begin{bmatrix} j\omega\mu_0\gamma_n \sinh \gamma_h d & j\omega\mu_0\gamma_0 & -\alpha\beta\frac{\epsilon_0}{\epsilon_2} \sinh \gamma_e d & \alpha\beta \\ 0 & 0 & \left(k_0^2 - \frac{\epsilon_0}{\epsilon_2}\alpha^2\right) \sinh \gamma_e d & (\alpha^2 - k_0^2) \\ -\alpha\beta \cosh \gamma_h d & \alpha\beta & -j\omega\epsilon_0\gamma_e \cosh \gamma_e d & -j\omega\epsilon_0\gamma_0 \\ (k_2^2 - \alpha^2) \cosh \gamma_h d & -(k_0^2 - \alpha^2) & 0 & 0 \end{bmatrix} \begin{bmatrix} A(\alpha) \\ B(\alpha) \\ C(\alpha) \\ D(\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\tilde{J}_x(\alpha, d) \\ \tilde{J}_z(\alpha, d) \end{bmatrix}. \quad (25)$$

Even though this matrix form appears to be simpler than the one for the previous case, the manipulations required to find the solution for the coefficients and the spectral domain impedance functions are more involved since the two TE and TM modes with respect to the optical axis are coupled for this orientation. The procedure, however, is straightforward, and solving (25) for $A(\alpha)$–$D(\alpha)$ and substituting in the expression for tangential electric field components at $y = d$ leads to the desired spectral domain impedance matrix where elements are found to be

$$\tilde{Z}_{xx}(\alpha, \beta, d) = \frac{(\alpha^2 - k_0^2)}{\Delta} g_{11} \quad (26a)$$

$$\tilde{Z}_{xz}(\alpha, \beta, d) = \tilde{Z}_{zx}(\alpha, \beta, d) = \frac{1}{\Delta}(k_0^2 - \alpha^2)F_2 = \frac{1}{\Delta}(\alpha\beta g_{11} - j\omega\mu_0\gamma_0 g_{12}) \quad (26b)$$

$$\tilde{Z}_{zz}(\alpha, \beta, d) = \frac{1}{\Delta}(j\omega\mu_0\gamma_0 F_1 - \alpha\beta F_2) \quad (26c)$$

where

$$g_{11} \triangleq j\omega\mu_0\left[\gamma_0 + \gamma_h \frac{(k_0^2 - \alpha^2)}{(k_2^2 - \alpha^2)} \tanh \gamma_h d\right]$$

$$g_{12} = g_{21} \triangleq \alpha\beta\left[1 - \frac{(k_0^2 - \alpha^2)}{(k_2^2 - \alpha^2)}\right]$$

and

$$g_{22} \triangleq -j\omega\epsilon_0\left[\gamma_0+\gamma_e\frac{\epsilon_2}{\epsilon_0}\frac{(k_0^2-\alpha^2)}{(k_2^2-\alpha^2)}\coth\gamma_e d\right]$$

$$\Delta \triangleq g_{11}g_{22}-g_{12}g_{21}$$

$$F_1 \triangleq \frac{\alpha\beta}{(k_2^2-\alpha^2)}g_{12}+\frac{j\omega\mu_0\gamma_h\tanh\gamma_h d}{(k_2^2-\alpha^2)}g_{22}$$

$$F_2 \triangleq \frac{\alpha\beta}{(k_2^2-\alpha^2)}g_{11}+\frac{j\omega\mu_0\gamma_h\tanh\gamma_h d}{(k_2^2-\alpha^2)}g_{21}.$$

The above impedance functions reduce to the ones for the isotropic medium case when $\epsilon_1=\epsilon_2=\epsilon$ if we formulate the isotropic medium problem in terms of TE and TM modes with respect to the $x$-axis. In addition, as $\omega\to 0$, the matrix leads to the static Green's function (from the corresponding affine transformation) as given by (22) with $\epsilon_1$ and $\epsilon_2$ interchanged.

### C. Optical Axis Along z-Direction ($\epsilon_x=\epsilon_y=\epsilon_2, \epsilon_z=\epsilon_1$)

For this case $\boldsymbol{\pi}_h=\pi_h\hat{a}_z$ and $\boldsymbol{\pi}_e=\pi_e\hat{a}_z$, where $\tilde{\pi}_e$ and $\tilde{\pi}_h$ are solutions of (in the spectral domain)

$$\frac{d^2\tilde{\pi}_{h1}}{dy^2}-\gamma_h^2\tilde{\pi}_{h1}=0$$

$$\frac{d^2\tilde{\pi}_{e1}}{dy^2}-\gamma_e^2\tilde{\pi}_{e1}=0, \qquad \text{in Region I}, y<d \qquad (27)$$

$$\frac{d^2\tilde{\pi}_{h2}}{dy^2}-\gamma_0^2\tilde{\pi}_{h2}=0$$

$$\frac{d^2\tilde{\pi}_{e2}}{dy^2}-\gamma_0^2\tilde{\pi}_{e2}=0, \qquad \text{in Region II}, y>d \qquad (28)$$

where for this case

$$\gamma_h^2 \triangleq \alpha^2+\beta^2-k_2^2$$

$$\gamma_e^2 \triangleq \alpha^2+\frac{\epsilon_1}{\epsilon_2}\beta^2-k_1^2$$

and

$$\gamma_0^2 \triangleq \alpha^2+\beta^2-k_0^2.$$

The procedure to find the spectral domain impedance matrix functions is very similar to the previous case (Section III-$B$) and also to the conventional approach used for the isotropic problem in terms of TE and TM modes with respect to the $z$-axis [13], [14]. The two modes in this case also are not decoupled. Writing the solution for fields in terms of unknown coefficients and applying the boundary conditions leads to the following impedance functions:

$$\tilde{Z}_{xx}(\alpha,\beta,d)=[j\omega\mu_0\gamma_0F_1-\alpha\beta F_2]/\Delta \qquad (29a)$$

$$\tilde{Z}_{xz}(\alpha,\beta,d)=\tilde{Z}_{zx}(\alpha,\beta,d)=(k_0^2-\beta^2)\cdot F_2/\Delta=-(j\omega\mu_0\gamma_0g_{12}+\alpha\beta g_{11})/\Delta \qquad (29b)$$

$$\tilde{Z}_{zz}(\alpha,\beta,d)=(k_0^2-\beta^2)g_{11}/\Delta \qquad (29c)$$

where

$$g_{11} \triangleq -j\omega\mu_0\left[\gamma_0+\gamma_h\frac{(k_0^2-\beta^2)}{(k_2^2-\beta^2)}\tanh\gamma_h d\right]$$

$$g_{12}=g_{21} \triangleq \alpha\beta\left[1-\frac{(k_0^2-\beta^2)}{(k_2^2-\beta^2)}\right]$$

$$g_{22} \triangleq j\omega\epsilon_0\left[\gamma_0+\gamma_e\frac{\epsilon_2}{\epsilon_0}\frac{(k_0^2-\beta^2)}{(k_2^2-\beta^2)}\coth\gamma_e d\right]$$

$$\Delta \triangleq g_{11}g_{22}-g_{12}g_{21}$$

$$F_1 \triangleq \frac{\alpha\beta}{(k_2^2-\beta^2)}g_{12}-\frac{j\omega\mu_0\gamma_h\tanh\gamma_h d}{(k_2^2-\beta^2)}g_{22}$$

and

$$F_2 \triangleq \frac{-\alpha\beta}{(k_2^2-\beta^2)}g_{11}+\frac{j\omega\mu_0\gamma_h\tanh\gamma_h d}{(k_2^2-\beta^2)}g_{21}.$$

Again, it is seen that for $\epsilon_1=\epsilon_2=\epsilon(\gamma_h=\gamma_e=\gamma)$ the above matrix functions reduce to the known expressions for the isotropic medium case [13]–[15] and lead to the spectral domain Green's function for ((22) with $\epsilon_1$ replaced by $\epsilon_2$) the static isotropic medium case having $\epsilon=\epsilon_2$ when $\omega\to 0$.

## IV. Concluding Remarks

It is shown that the propagation characteristics of inhomogeneous guided wave structures with uniaxial dielectric media can be studied in a unified convenient manner by utilizing the auxilliary electric and magnetic Hertzian potential functions having components along the optical axis only. This corresponds to formulation of the boundary value problem in terms of the TE and TM modes with respect to the optical axis. For the case of the optical axis normal to the interface, the two modes (also called LSM and LSE) are decoupled. In order to illustrate the procedure, the derivation and results are presented for a single layer uniaxial medium with a ground plane for all three orientations of the optical axis. The spectral domain immittance functions required to solve for the dispersion characteristics of other planar structures are found by following the same steps. Other propagation characteristics such as characteristic impedance is then found in terms of the fields and the phase constant. No attempt was made to include the numerical results in this paper since the computation methods used to solve (21), (26), and (29) for the dispersion characteristics are the same as those for the isotropic medium case [13], [17].

It should be noted that even though the main emphasis of the paper was the formulation of the eigenvalue problem for general planar waveguide structures, the procedure given in Section II is a general one and can be applied to various waveguides and other boundary value problems with uniaxial media.

## References

[1] R P. Owens, J. E. Aitken, and T. C. Edwards, "Quasi-static characteristics of microstrip on an anisotropic sapphire substrate,"

*IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 499–505, Aug. 1976.

[2] T. C. Edwards and R. P. Owens, "2–18 GHz dispersion measurements on 10–100Ω microstrip line on sapphire," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 506–513, Aug. 1976.

[3] B. T. Szentkuti, "Simple analysis of anisotropic microstrip lines by a transform method," *Electron. Lett.*, vol. 12, pp. 672–673, Dec. 1976.

[4] E. Yamashita, K. Atzuki, and T. Mori, "Application of MIC formulas to a class of integrated-optics modulator analysis: A simple transformation," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 146–150, Feb. 1977.

[5] M. Kobayashi, "Green's function technique for solving anisotropic electrostatic field problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 510–512, June 1978.

[6] M. Kobayashi and R. Terakado, "New view on an anisotropic medium and its application to transformation from anisotropic to isotropic problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-27, pp. 769–775, Sept. 1979.

[7] N. G. Alexopoulos and C. M. Krowne, "Characteristics of single and coupled microstrips on anisotropic substrates," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 387–393, June 1978.

[8] M. Kobayashi and R. Terakado, "Method for equalizing phase velocities of coupled microstrip lines by using anisotropic substrate," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 719–722, July 1980.

[9] M. Horno, "Quasistatic characteristics of microstrip on arbitrary anisotropic substrates," *Proc. IEEE*, vol. 86, pp. 1033–1034, Aug. 1980.

[10] M. Kobayashi, "Analysis of the microstrip and the electrooptic light modulator," *IEEE Trans. Microwave Theory Tech.*, MTT-26, pp. 119-126, Feb. 1978.

[11] K. R. Carver and J. W. Mink, "Microstrip antenna technology," *IEEE Trans. Antenna and Propagation*, vol. AP-29, pp. 2–24, Jan. 1981.

[12] E. Yamashita, K. Atsuki and T. Akamatsu, "Application of microstrip analysis to the design of a broadband electrooptic light modulator," *IEEE Trans. Microwave Theory and Tech.*, MTT-22, pp. 462–464, Apr. 1974.

[13] T. Itoh and R. Mittra, "A technique for computing dispersion characteristics of shielded microstrip lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp. 896–898, Oct. 1974.

[14] R. Mittra and T. Itoh, "*Analysis of microstrip transmission lines*," in *Advances in Microwaves*, vol. 8, L. Young and H. Sobol, Eds. New York: Academic, 1974, pp. 67–141.

[15] J. B. Knorr and K. D. Kuchler, "Analysis of coupled slots and coplanar strips on dielectric substrate," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 541–547, July 1975.

[16] J. B. Davies and D. Mirshekar-Syahkal, "Spectral domain solution of arbitrary coplanar transmission line with multilayer substrate," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 143–146, Feb. 1977.

[17] R. H. Jansen, "High speed computation of single and coupled microstrip parameters including dispersion, higher order modes, loss and finite strip thickness," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 75–82, Feb. 1978.

[18] T. Itoh, "Spectral domain immittance approach for dispersion characteristics of generalized printed transmission lines," *IEEE Trans. Microwave Theory and Tech.*, vol. MTT-28, pp. 733–736, July 1980.

[19] A. A. Th. M. Van Trier, "Guided electromagnetic waves in anisotropic media," *Applied Sci. Res.*, vol. B3, pp. 395–371, 1953.

[20] R. E. Collin, *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960, pp. 101–102.

[21] N. S. Kapany and J. J. Burke, *Optical Waveguides*. New York: Academic, 1972, ch. 4, sect. E.

[22] E. J. Denlinger, "A frequency dependent solution for microstrip transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 30–39, Jan. 1971.

**Hyuckjae Lee** (S'77) was born in Kyunggi-do, Korea, in 1947. He received the B.S. degree in electronics engineering from Seoul National University, Seoul, Korea, in 1970, and is now completing the Ph.D. requirements at Oregon State University, Corvallis.

After graduation from college, he served his military duty as a naval engineering officer. He is presently engaged in thesis work on planar MIC structure characteristics. His research interests are computational applications of electromagnetic field problems, microwave solid state devices, and computer software developments.

**Vijai K. Tripathi** (M'68) received the B.Sc. degree from Agra University, Uttar Pradesh, India, the M.Sc. Tech. degree in electronics and radio engineering from Allahabad University, Uttar Pradesh, India, and the M.S.E.E., and Ph.D. degrees in electrical engineering from the University of Michigan, Ann Arbor, in 1958, 1961, 1964, and 1968, respectively.

From 1961 to 1963 he was a Senior Research Assistant at the Indian Institute of Technology, Bombay, India. In 1963 he joined the Electron Physics Laboratory of the University of Michigan where he worked as a Research Assistant from 1963 to 1965, and as a Research Associate from 1966 to 1967 on microwave tubes and microwave solid-state devices. From 1968 to 1973 he was an Assistant Professor of Electrical Engineering at the University of Oklahoma, Norman. In 1974 he joined Oregon State University, Corvallis, where he is an Associate Professor of Electrical and Computer Engineering. He has been on sabbatical leave during the 1981–82 academic year and was with the Division of Network Theory at Chalmers University of Technology in Gothenburg, Sweden, from November 1981 through May 1982. His current research activities are in the areas of microwave circuits and devices, electromagnetic fields, and solid-state devices.

Dr. Tripathi is a member of Eta Kappa Nu, Sigma Xi, and the American Society for Engineering Education.